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THE INTAKE OF PURIFIED FRESH WATER WITH FILTRATION FROM A RESERVOIR*

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A planar formulation is used to study the filtration flow of fresh water above saline water, in the general case when the flow consists of several streams moving in different directions. The problem is reduced to the study of a second-order linear differential equation of Fuchs class, with four regular singularities, whose integration encounters, as shows in /1, 2/, major mathematical difficulties. In order to construct the integrals and determine the unknown constants appearing in the corresponding Fuchs equation, it is proposed to employ the method described in /3/, and the solution of the problem is given in closed form in terms of elementary functions. The solution obtained is used to derive the algorithm for computing the size of the lens, and the influence of all factors governing the flow on its filtration characteristics is analysed. In addition to the special case /4/ in which the stream line separating the drained and evaporating flows within the lens passes into the lower corner point of the line of separation, limit cases related to the "solidification" of saline water and the lack of drainage /5-8/ are also discussed.

1. Formulation of the problem. Fig.1 shows schematically the right-hand half of the transverse cross-section of the fresh water lens formed in the homogeneous and isotropic soil above the saline waters at rest, under the condition of steady filtration from a reservoir or canal. The reservoir is modelled by a horizontal segment of length $2l$, and the depth of water in the reservoir is assumed to be infinitely small. Let one part of the flow passing from the reservoir to the soil evaporate from the free surface of the lens at a constant rate ε referred to the soil filtration coefficient $\varkappa$), and let the other part flow through a horizontal drain of diameter D, inserted in the plane of symmetry of the reservoir, at a depth S below it. The intake of fresh purified water is, under these conditions, of well-known practical interest.

Let us introduce the complex potential of the flow $\omega = \varphi + i\psi$ and the complex coordinate of the points belonging to the domain of motion $z = x + iy$ referred, respectively, to $\varkappa T$ and T, where T is the depth of the horizontal surface of saline water outside the lens, measured from the plane $y = 0$. During the initial investigation we replace the drain by a point sink situated at the point M.

The problem consists of determining the depression curve CD and the line of separation BC under the following boundary conditions:

$$\begin{gathered} AM: x = 0,\ \psi = 0;\ MB: x = 0,\ \psi = Q_d \\ AD: y = 0,\ \varphi = 0,\ BC: \varphi - \rho y = T(1+\rho),\ \psi = Q_d \\ CD: \varphi + y = 0,\ \psi + \varepsilon x = Q_c/2 + \varepsilon l \\ (Q_d = Q_c/2 - Q_e,\ Q_e = \varepsilon(L - l),\ \rho = \rho_2/\rho_1 - 1 > \varepsilon) \end{gathered} \tag{1.1}$$

Here Q_c and Q_d are the corresponding filtration flows from the reservoir (per unit length) and the drain, Q_e is the value of the total evaporation from the free surface of the lens within the region in question, and ρ_1 and ρ_2 are the densities of the fresh and saline water respectively. In addition to determining the size of the lens (width $2L$, maximum depth H_1 and the depth H at the intersection by the plane $x = 0$), we also determine the flows Q_c, Q_d and Q_e.

In order to solve the problem we introduce the auxiliary variable $\xi = \zeta + i\eta$, and the functions: $z(\xi)$, which maps conformally the upper semiplane ξ onto the region z (the correspondence of the points is shown in Fig.2a), the complex velocity $w = d\omega/dz$, and the functions

$$F(\xi) = d\omega/d\xi,\ Z(\xi) = dz/d\xi \tag{1.2}$$

2. Integrating the Fuchs-class equations with four singularities. Fig.3 shows the region of complex velocity corresponding to the boundary conditions (1.1) and representing a circular quadrangle. The region is characterized by the presence of a cut, whose configuration depends on the position of the central point N, i.e. of the point of zero velocity at the boundary of the domain of motion. The circular cut BGC corresponds to the case when the point of separation of the flows lies on the line of separation BC (the dashed in Fig.3) when that point arrives at the segment BM.

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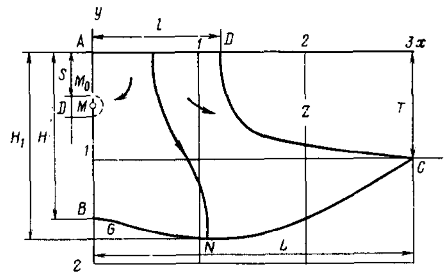
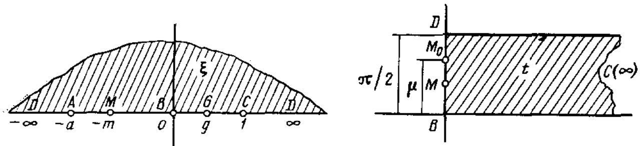


Fig.1



a

b

Fig.2

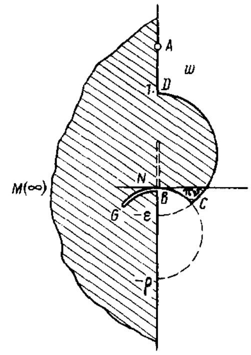


Fig.3

In order to map conformally the given circular quadrangle onto a half-plane, we must construct two integrals of the following second-order linear differential equations:

$$v'' + \left(\frac{1}{2\xi} - \frac{1}{\xi - g} + \frac{1-\nu}{\xi - 1}\right)v' - \frac{(1-\nu)(2+\nu)\xi - \lambda}{4\xi(\xi - g)(\xi - 1)}v = 0 \tag{2.1}$$

of Fuchs class, with four regular singularities. We know /1, 2/ that difficulties are encountered when integrating such equations. The difficulties are caused by the fact that the coefficients of Eq.(2.1) contain, apart from the undetermined quantity g, also a so-called accessory parameter λ, which is also not known in advance. These parameters, which cannot be fully determined from the geometrical characteristics of the quadrangle, must be determined simultaneously when constructing the integrals, but there is as yet no method of solving this problem which is sufficiently general and convenient. We must therefore resort to various indirect methods, and here we must mention, before anything else, the papers /9-11/ which use the problem of linear conjugation.

We shall show that in the present case Eq.(2.1) allows a direct determination of particular solutions expressed in terms of elementary functions.

We will make the following change of variables in Eq.(2.1):

$$\xi = \operatorname{th}^2 t \tag{2.2}$$

which maps the upper half-plane ξ onto the half-strip $u > 0,\ 0 < v < \pi/2$ of the plane $t = u + iv$ (Fig.2b), and following /3/ we shall seek two linearly independent solutions of the equation obtained in the form

$$v_{1,2} = \operatorname{ch} t^{-2-\nu}\left[(C_1 \operatorname{ch}^2 t + C_2)\begin{Bmatrix}\operatorname{ch} \nu t\\ \operatorname{sh} \nu t\end{Bmatrix} + {}^1\!/_2 \nu C_2 \operatorname{sh} 2t \begin{Bmatrix}\operatorname{sh} \nu t\\ \operatorname{ch} \nu t\end{Bmatrix}\right] \tag{2.3}$$

Here we take the upper expression within the braces for the first integral, and the lower expression for the second integral, C_1 and C_2 are certain constants which do not vanish simultaneously, and $C_1 + 2C_2 \neq 0$ and $3C_1 + 2C_2 + C_2\nu^2 \neq 0$. We can confirm by direct substitution that the result of substituting (2.3) into the transformed Eq.(2.1) vanishes identically, provided that the following two conditions hold:

$$\begin{gathered}[C_1\nu(\nu - 1) + C_2(3\nu^2 - \nu - 2)]\,g + (C_1 + C_2)\,\lambda = 0\\ [C_1\nu(\nu - 1) + 2C_2(\nu^2 + 2)]\,g + C_1\lambda = 2\,(C_1 + 2C_2)\end{gathered} \tag{2.4}$$

Using system (2.4), whose determinant is not zero under the conditions given above imposed on the constants C_1 and C_2, we find the required parameters in the form

$$g = 1 - \frac{C_2^2\nu^2 - C_1^2}{C_2(3C_1 + 2C_2 + C_2\nu^2)}, \quad \lambda = \frac{C_1(2 + g\nu - g\nu^2) + C_2(4 - 4g - 2g\nu^2)}{C_1} \tag{2.5}$$

We note that if we use Liouville's formula /2/ $(v_2/v_1)' = C\exp\left(-\int p(\xi)\,d\xi\right)/v_1^2$, where $p(\xi)$

is the coefficient of v' in (2.1) and use (2.3), we again arrive at the relation (2.5). If on the other hand we regard (2.4) as a system in C_1 and C_2 with g and λ given, then in order for the homogeneous system to have a non-zero solution it is necessary and sufficient that its determinant be zero

$$\lambda^2 - 2\lambda\,[g\,(3+\nu-\nu^2)-1] + g\,(1-\nu)(2+\nu)[g\nu\,(3-\nu)+2] = 0 \tag{2.6}$$

It is interesting to note that (2.6) is identical with the well-known condition of Polubarinova-Kochina for the point G which represents the tip of the cut /6/.

3. ***Constructing the function*** w, F ***and*** Z**.** A function which maps conformally a half-strip of the t plane onto a given circular quadrangle of the w plane must be expressed as the ratio of linear combinations of the solutions v_1 and v_2. If we construct such combinations and use the correspondence of the points B, C and D in the t and w planes, we obtain

$$w = \frac{F}{Z} = \gamma\rho\,\frac{v_2}{v_1 + i\gamma v_2} \tag{3.1}$$
$$\gamma = \sqrt{\varepsilon/[(\rho+1)(\rho-\varepsilon)]},\quad \nu = 2\pi^{-1}\operatorname{arcctg}\sqrt{\varepsilon\,(\rho+1)/(\rho-\varepsilon)}$$

We will determine the functions F and Z using Polubarinova-Kochina's method /1/. Taking into account relations (3.1) and (2.2) we obtain

$$d\omega/dt = C\,[(\operatorname{ch}^2 t + A)\operatorname{sh}\nu t + {}^1/_2\nu A\operatorname{sh} 2t\operatorname{ch}\nu t]/\Delta \tag{3.2}$$
$$dz/dt = C\,(\gamma\rho)^{-1}\,[(\operatorname{ch}^2 t + A)(\operatorname{ch}\nu t + i\gamma\operatorname{sh}\nu t) + {}^1/_2\nu A\operatorname{sh} 2t\,\times$$
$$(\operatorname{sh}\nu t + i\gamma\operatorname{ch}\nu t)]/\Delta,\quad \Delta = (m + \operatorname{sh}^2 t)\sqrt{a + \operatorname{sh}^2 t}$$
$$C = C_1 > 0,\quad A = \frac{C_2}{C_1} = (1-m)\left[\nu\sqrt{m\,(1-m)}\,\frac{\gamma+\delta}{1-\gamma\delta} - 1\right]$$
$$\delta = \operatorname{tg}\nu\arcsin\sqrt{m}.$$

Here A is a constant controlling the position of the cut in the w plane. When $A < -\nu^{-1}$, $-1 < A < -{}^1/_2$ or $\nu^{-1} < A$, we have a circular cut, for $-3/(2+\nu^2) < A < -1$ or $-{}^1/_2 < A < 0$ we have a vertical cut, and when $A = -{}^1/_2$ the cut disappears and the circular quadrangle degenerates into a triangle.

It can be confirmed that the functions (1.2) defined with help of relations (3.2) and (2.2) satisfy the boundary conditions (1.1) written in terms of these functions, and represent therefore a parametric solution of the initial problem.

4. ***Computing the flow pattern.*** Eqs.(3.2) serve as the solution of the problem for a point sink. Let us extend the results obtained to the case of a drain pipe of small, nearly semicircular cross-section. To do this we take, as the contour, the line of equal pressure passing through the upper point M_0 of the cross-section of the pipe of diameter D, and we denote by μ the ordinate of this point in the t plane. We find that, as a result, Eq.(3.2) will contain four unknown constants: C, a, m and μ, and we use in their determination the width of the reservoir $2l$, the pipe diameter D, the depth of its insertion S, and the depth T of the horizontal surface of saline water outside the lens.

Eqs.(3.2), which can be integrated over various segments of the boundary of the region t, lead to parametric equations for the corresponding sections of the scheme. We confirm the monotonic form of the functions in these equations using numerical methods and establish in this manner the unique solvability of the system relative to the unknown constants

Table 1

l	$2L$	H_1	Q_c	$\delta\cdot 10^2$	ρ	$2L$	H_1	Q_c	$\delta\cdot 10^2$
0.3	4.49	1.87	0.59	42	0.15	6.67	2.62	2.59	39
0.6	5.80	2.09	1.28	36	0.18	6.51	2.42	2.61	37
0.9	6.07	2.01	2.23	33	0.22	6.34	2.21	2.63	35
1.2	6.02	1.77	3.60	29	∞	4.79	1.00	2.73	21

Table 2

$\varepsilon\cdot 10^2$	$2L$	H_1	Q_c	$\delta\cdot 10^2$	$D\cdot 10^2$	$2L$	H_1	Q_c	$S\cdot 10^2$	$2L$	H_1	Q_c
9.5	6.20	1.93	2.63	31	5	6.33	2.02	2.89	8	6.84	2.17	5.49
10	6.06	1.92	2.62	32	10	6.23	1.98	2.82	20	6.64	2.11	3.68
15	5.13	1.90	2.59	37	30	5.90	1.87	2.51	35	6.23	1.98	2.81
25	4.23	1.87	2.54	44	40	5.73	1.82	2.43	50	5.73	1.82	2.44

Having found the required parameters, we determine the dimensions $2L, H_1$ and H of the lens, the pressure h_0, in the drain, and the flow rates Q_c, Q_d and Q_e, and find the coordinates of the points of the depression curve and free surface. Fig.1 depicts a lens computed for $T = 1$; $l = 1.2$; $D = 0.2$; $S = 0.4$; $\varepsilon = 0.1$ and $\rho = 0.3$.

5. *Special and limiting cases.* We shall mention certain special and limiting cases connected with the degeneration of the complex velocity region and with the limiting values of the mapping parameters.

A. When $A = -1/2$, the circular quadrangle degenerates as was shown above, to a triangle. In the z plane this value of A corresponds to the special case of a flow in which the stream line dividing the draining and evaporating flows within the lens arrives at the point B, i.e. when the points B and N coincide /4/.

B. When $\rho = \infty$, i.e. $\rho_2 = \infty$, we have a case which can be treated within the framework of the filtration scheme used here and "congealment" of saline water. The formulas obtained lead to a solution of the problem of the inflow of fresh water to the drain, when there is no separation bounary, but a water-confining stratum exists at a finite depth. If in addition $l = \infty$, then the ground surface becomes completely flooded. This case was first studied in /12/. Integrating Eqs.(3.2) with $\varepsilon = 0$ we obtain

$$z = T\,(2\pi^{-1}t - i) \qquad (5.1)$$
$$\omega = Q_d\,[{}^1/{}_2 i + \pi^{-1}\operatorname{arth}(\sqrt{1-m}/\operatorname{ch} t)]$$

and we find that $\mu = {}^1/{}_2\pi T^{-1}(T - S - {}^1/{}_2 D)$. From (5.1) we can obtain a well-known formula /13/ connecting the filtration flow rate Q_d through the drain with the pressure head h_0 along the drain contour.

C. When there is no drain and all filtration flow from the reservoir goes to evaporation, the region of complex velocity also degenerated into a triangle. In this case the solution of the problem /5-7/ can be obtained from relations (3.2) for $m = 1, A = 0$. If in addition we have $\rho = \infty$, then we have the special case which was discussed in /14/.

6. *Assessment of numerical results.* Tables 1 and 2 give the results of computing the filtration characteristics, based on clarifying the influence of the parameters l, D, S, ε and ρ on the flow pattern. Only one of the parameters is varied in each section of the table, and the remaining parameters are fixed for the values $l = 1, D = 0.2, S = 0.4, \varepsilon = 0.1$ and $\rho = 0.3$. Moreover, the quantity $\delta = H_1/(2L)$ /7/ appears in the table reflecting, to a certain degree, the form of the lens. When ε is varied, the quantity characterizes the degree of elongation of the lens, and for fixed ε it expresses its degree of compression. The results enable us to draw certain conclusions concerning the effect of the physical parameters on the form and size of the lens, and also on the character of the dependence of filtration flow rates Q_c and Q_d.

The qualitative agreement between the results when the parameters $l\,(l \geqslant 0.9)$ and ρ are varied in Table 1 and the parameters ε, D and S in Table 2, merits our attention. When the values of ρ/ε are equal, the quantity δ remains practically unchanged, i.e. reducing the parameters ε and ρ the same number of times leads to a uniform increase in the size of the lens. For example, for $\rho/\varepsilon = 3$ we have $\delta = 0.32$.

Table 2 shows that the diameter of the drain and depth of its placement have no effect on the size of the lens, since they lead to insignificant changes in the values of $2L$ and H_1 (by a factor of $1.1-1.2$), and this in turn affects the quantity δ only slightly. The intensity of evaporation ε, and the width of the reservoir exert the greatest influence on the lens width. Thus from Table 1 it follows that when the value of ε is reduced from 0.25 to 0.095, the lens width will increase by 47%, and by 35% when l increases from 0.3 to 0.9. The greatest changes in the depth of the lens H_1 are observed when the parameter ρ is varied; when its value is reduced, i.e. when the head is reduced on the side of saline water from 0.25 to 0.15, the magnitude of H_1 will increase by more than 26%.

The first section of Table 1 shows that the dependence of the lens size on the width of the reservoir is not monotonic: the quantities $2L$ and H_1 take their maximum values 6.0697 and 2.0914 for values of l equal to 0.9 and 0.7, respectively.

Computations have shown that for the given combination of physical parameters the point N at which the flows separate falls at the line of separation BC for the following values of the physical parameters: $l > 1, \rho > 0.3, 0 < \varepsilon < 0.15, D > 0.1$ and $S > 0.3$. It is clear that when the point N lies on the segment BM, when $H = H_1$.

The analysis of the dependence of the filtration flow rates Q_c and Q_d on the defining parameters of the scheme reduces to the following. The width of the reservoir exerts the greatest influence on these quantities. When l is increased four-fold the flow rates Q_c and Q_d increase by 6.1 and 16 times respectively, and the flow Q_c from the reservoir may exceed the flow through the drain Q_d by 6 or more times for small values of $l\,(l < 0.5)$. At these values of l the amount of water lost by evaporation from the free surface may also exceed the amount of water absorbed by the drain. Thus for $l = 0.3$ we have $Q_e = 0.194$ and $Q_d = 0.101$, and hence $Q_c \approx 2Q_d$. When the width of the reservoir is increased further, corresponding to an insignificant increase in the value of Q_e, we have a significant increase in the drainage. When $l = 1.2$, we have $Q_e = 0.181$ and $Q_d = 1.62$, i.e. we now have $Q_d \approx 9Q_c$.

We note that for the values of the parameters ρ, ε, D and S given in Table 1 and 2, we have the approximate equality $Q_c \approx 2.4\,Q_d$.

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